11-5-21

One last spherical Example

Ex: compute the volume of closed disk of radius $\alpha > 0$

NB: done in cartesian coordinates but it was nasty

$$D_\alpha = \{(\rho, \theta, \varphi) : 0 \le \rho \le \alpha, 0 \le \theta \le 2\pi, 0 \le \varphi \le \pi\}$$

$$\text{Vol}(D_\alpha) = \iiint_{D_{\alpha\,\text{cart}}} 1\, dV$$

$$= \iiint_{D_{\alpha\,\text{sphere}}} 1 \cdot \rho^2 \sin(\varphi)\, dV_{\text{sphere}}$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \rho^2 \sin(\varphi)\, d\varphi\, d\theta\, d\rho$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} -\rho^2 [\cos\varphi]_{\varphi=0}^{\pi}\, d\theta\, d\rho$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} -\rho^2(-1-1)\, d\theta\, d\rho$$

$$= 2 \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} \rho^2\, d\theta\, d\rho = 2 \int_{\rho=0}^{\alpha} \rho^2 [\theta]_{\theta=0}^{2\pi}\, d\rho = 4\pi \int_{\rho=0}^{\alpha} \rho^2\, d\rho$$

$$= \frac{4}{3}\pi(\alpha^3 - 0) = \frac{4}{3}\pi\alpha^3$$

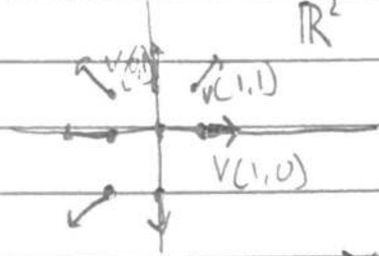
## § 16.1: Vector Fields

Goal: Study functions $\vec{V}: \mathbb{R}^n \to \mathbb{R}^n$
(most of time $n=2$ or $n=3$)

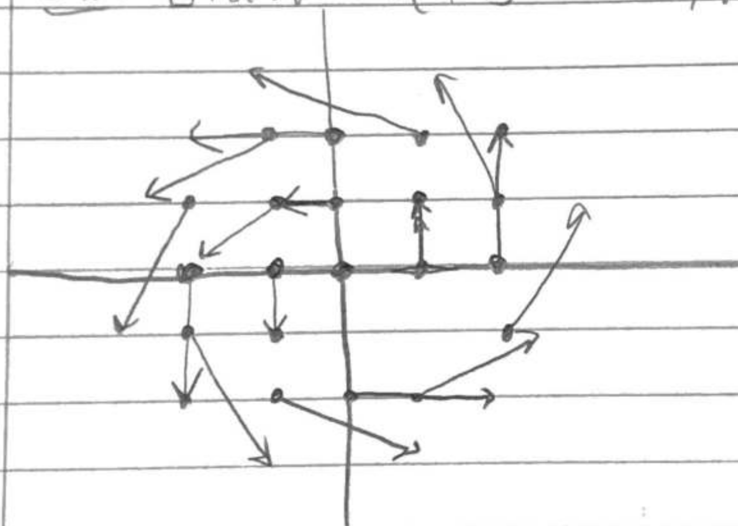
Vector field - is a function $\vec{v}: \mathbb{R}^n \to \mathbb{R}^n$

$\vec{V}(x, y) = \langle x, y \rangle$ on $\mathbb{R}^2$

every point in $\mathbb{R}^2$ has an associated vector attatched to it by this vector field.




Ex: Draw $\vec{V}(x, y) = \langle -y, x \rangle$



$V(0,0) = \langle 0,0 \rangle$
$V(1,0) = \langle 0,1 \rangle$
$V(2,0) = \langle 0,2 \rangle$
$V(-1,0) = \langle 0,-1 \rangle$
$V(-2,0) = \langle 0,-2 \rangle$
$V(-1,1) = \langle -1,-1 \rangle$
$V(0,1) = \langle -1,0 \rangle$
$V(-2,1) = \langle -1,-2 \rangle$

Ex: The gradient of a function is always a vector field
↳ e.g. for $f(x,y) = xy$ $\nabla f = \langle y, x \rangle$ is a vector field on $\mathbb{R}^2$. Vector field = "vf"

A vector field like this is sometimes called a "gradient vector field

e.g. $f(x,y,z) = e^{x+y^2}\cos(z+x)$

$\nabla f = \langle e^{x+y^2}\cos(z+x) - e^{x+y^2}\sin(z+x),\ 2y\cos(z+x)e^{x+y^2} - e^{x+y^2}\sin(z+x)\rangle$

is a vector field $\vec{\nabla} f(x,y,z) =$

Obvious Question: How do we know when a vector field is a gradient vector fields?

↳ is every v.f. a grad v.f.?

Terminology: A conservative vector field is a gradient v.f.

If $\vec{V} = \nabla f$ is a conservative v.f., we call $f$ a potential function for $\vec{V}$

Now, is every v.f. conservative?

On $\mathbb{R}^2$, a conservative v.f. has form $\vec{v} = \langle f_x(x,y), f_y(x,y)\rangle$ for some potential function $f$. By Clairaut's Theorem, $f_{xy} = f_{yx}$, so every conservative v.f. $\vec{v} = \langle \alpha(x,y), \beta(x,y)\rangle$ has to satisfy

$$\alpha_y = \beta_x$$

$$V_x \quad V_y$$

Ex: $\vec{V} = \langle -y, x \rangle$

$\frac{\partial}{\partial y}[V_x] = \frac{\partial}{\partial y}[-y] = -1$ $\vec{V}$ is non conservative it violates Clairaut's Theorem.

$\frac{\partial}{\partial x}[V_y] = \frac{\partial}{\partial y}[x] = 1$

$\therefore$ Not every vector field is a gradient vector field !!

Prop: A vector field $\vec{v} = \langle V_{x_1}, V_{x_2}, \dots, V_{x_n} \rangle$ is conservative if and only if $\frac{\partial}{\partial x_i}\left[V_{x_j}\right] = \frac{\partial}{\partial x_j}\left[V_{x_i}\right]$

for all $i,j$ (i.e. a vf is conservative if and only if it satisfies Clairaut's theorem)

Ex: Is $\vec{v} = \langle x, y \rangle$ conservative?

Sol: $\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}[x] = 0$

$\frac{\partial V_y}{\partial x} = \frac{\partial}{\partial x}[y] = 0$

$\therefore$ by the proposition $\vec{v} = \langle x, y \rangle$ is conservative
what is it's potential function?

By conservativity, $\nabla f = \vec{v}$ for function $f(x,y)$
i.e. $f_x(x,y) = x$ and $f_y(x,y) = y$.

B/c $\frac{\partial f}{\partial x} = x$, we know $f(x,y) = \int \frac{\partial f}{\partial x} dx = \int x\, dx = \frac{1}{2}x^2 + C(y)$

$$\therefore y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1}{2}x^2 + C(y)\right] = \frac{dC}{dy}$$

honest constant

$$\therefore C(y) = \int \frac{dC}{dy} dy = \int y\, dy = \frac{1}{2}y^2 + D$$

$f(x,y) = \frac{1}{2}x^2 + C(y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + D$ for any constant

D is a potential function of $\vec{V}$.

Ex: $\vec{V} = \langle 2xy, x^2 - 3y^2 \rangle$ conservative? if yes, potential?

Sol:

$$\frac{\partial V_y}{\partial x} = \frac{\partial}{\partial x}\left[x^2 - 3y^2\right] = 2x$$

$$\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}\left[2xy\right] = 2x$$

$\vec{V}$ is conserved

$$f = \int f_x\, dx = \int 2xy\, dx = x^2 y + C(y)$$

$f_y = \frac{\partial}{\partial y}\left[x^2 y + C(y)\right]$ D → honest constant

$x^2 - 3y^2 = x^2 + C'(y)$

$C'(y) = -3y^2$ $C(y) = \int -3y^2 dy = -y^3 + D$

$f(x,y) = x^2 y + C(y) = x^2 y - y^3 + D$

NB: the method for computing the potentials can be used to prove the proposition from earlier...

Ex: $\vec{V} = \langle \ln(x+y), e^{x+y} + \frac{1}{x+y} \rangle$
conservative? If yes, potential?

Sol: $\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}[\ln(x+y)] = \frac{1}{x+y}$ $\cancel{\neq}$

$\frac{\partial V_y}{\partial x} = \frac{\partial}{\partial x}\left[e^{x+y} + \frac{1}{x+y}\right] e^{x+y} - (x+y)^{-2}$

Not conservative. No potential

Last time: you saw

$$\int_{y=c}^{d}\int_{x=a}^{b} f(x)\, g(y)\, dx\, dy = \int_{x=a}^{b} f(x)dx \cdot \int_{y=c}^{d} g(y) dy$$

true only when
① integrating over rectangle $[a,b] \times [c,d]$
② Integrand is a "separable function"
i.e. $h(x,y) = f(x) \cdot g(y)$